

# A classical solution in SU(2) Yang-Mills gauge theory

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August 23, 2011

## Abstract

A solution of euclidean Yang-Mills gauge theory, which is governed by  $\pi_4(SU(2))$ , is given.

## 1 Introduction

A new classical solution of the SU(2) Yang-Mills gauge theory is given. It is based on the homotopy group  $\pi_4(SU(2)) = \pi_4(S^3) = Z_2$ . Various ansätze are discussed, either self-dual as in the case of instantons [8] with finite action but zero energy or in Minkowski space with finite energy as in the meron case [3]. We will also investigate whether the Corrigan-Fairlie ansatz works, i.e., finding a  $\phi^4$  solution and plugging it into the general ansatz given by them [7].

## 2 Main part

In four dimensional euclidean Yang-Mills gauge theory, the base space (referring to the language of principal fiber bundles)  $R^4$  can be compactified to the fourdimensional sphere  $S^4$  because pure Yang-Mills gauge theory is conformally invariant and the metric of the fourdimensional sphere is conformally flat:

$$g_{\mu\nu} = \Omega(x)\eta_{\mu\nu}$$

where  $g_{\mu\nu}$  is the metric of the sphere  $S^4$ ,  $\eta_{\mu\nu}$  the metric of  $R^4$  (flat metric) and  $\Omega$  the conformal transformation. [9] Since we look at mappings  $g$  from the base space of the principal fiber bundle

into its structure group  $SU(N)$  (say), they are classified by  $\pi_4(SU(N))$  which is equal to  $Z_2$  in the case  $N = 2$ , otherwise 0. [6] In the case of instantons [4] and merons one employs boundary conditions, that enforce mappings from the equator of the base space  $S^4$ , that is  $S^3$ , to  $SU(N)$ . These are classified by  $\pi_3(SU(N)) = \pi_3(SU(2)) = \pi_3(S^3) = Z$  because of Bott periodicity (for instance D.Husemoller, fiber bundles ) The solutions are then classified by elements of  $\pi_3(S^3) = Z$ , which is called their topological charge  $k$ . In the case of instantons, selfduality of the curvature tensor  $F_{\mu\nu}$  is demanded, to have finite action solutions (and zero energy) , because it is believed, that these dominate the euclidean path integral in the case of semiclassical calculations. [5] The instanton solution with  $k = 1$  is given by

$$A_\mu = \frac{i\sigma_{\mu\nu}x_\nu}{x^2 + \lambda^2} g^{-1}\partial_\mu g$$

where  $\lambda$  is the scaling factor (pure Yang-Mills gauge theory is scale invariant and for merons with  $k = 1$

$$A_\mu = \frac{1}{2} g^{-1}\partial_\mu g$$

(due to the factor  $1/2$  this is not a pure gauge, hence  $F_{\mu\nu}$  is not zero, and gives a solution with finite energy [3]). Besides, the one form  $g^{-1}\partial_\mu g$  is the pullback of an element of the vertical subspace of the tangentspace of the bundle space  $P$ .

(We can generalize the factor  $\frac{1}{2}$  to  $\frac{m}{n}$  with  $m \neq n$  ) The gauge transformations  $g$  are representants of  $\pi_3(S^3)$ , and hence mappings from  $S^3$  to  $S^3$  with mapping degree 1 while  $g^{-1}\partial_\mu g$  is the pullback (see Bott, Differential forms in Algebraic Topology) Explicit formulas for the representants shall be given in a forthcoming paper. To find first solutions one can replace the  $[g]$  of  $\pi_3(S^3)$  with  $[g]$  out of  $\pi_4(S^3)$  in the instanton and meron solution ( $[g]$  means equivalence class of the mapping  $g$ ) . These have then topological charge out of  $Z_2$  Further solutions can be found by employing the theorem of Corrigan and Fairlie [7]:  $A_\mu = i\sigma_{\mu\nu}\partial_\nu \ln\phi$  is a solution of the Yang-Mills gauge theory, if  $\phi$  is a solution of the  $\phi^4$  theory. There is a relationship between  $\pi_4(S^3)$  and braid groups. [2] Also there is a lot to be said about  $\pi_4(SU(2))$  itself [6] These two items and further solutions shall be investigated in a forthcoming paper. An explicit, nontrivial representant of the homotopy group  $\pi_4(SU(2))$  can be found by suspending the Hopf-map, i.e., a representant of  $\pi_3(SU(2))$ . [1] Although  $\pi_4(SU(N))$  is zero for  $N$  larger than 2, the author<sup>1</sup> wonders whether there is any connection to the confinement problem and triality ( $Z_3$  vortices) .

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## References

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